International Journal of Theoretical Physics, Vol. 19, No. 12, 1980

Quadratic Lagrangians and the Reissner–Nordström–de Sitter Metric

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Received April 21, 1980

The purpose of this note is to point out that the Einstein–Maxwell equations with cosmological constant can be derived from the quadratic Lagrangians R^2 and $F_{\mu\nu}F^{\mu\nu}$. The linear combination $R_{\alpha\beta}R^{\alpha\beta}+\beta R^2+k_2F_{\mu\nu}F^{\mu\nu}$ leads to field equations not satisfied by the Reissner–Nordström–de Sitter metric.

1. INTRODUCTION

As is well known, the Riemann scalar density $R(-g)^{1/2}$ is not invariant with respect to the Weyl gauge group (Stephenson, 1977) whereas the the scalar densities

$$R^2(-g)^{1/2},\ R_{\alpha\beta}R^{\alpha\beta}(-g)^{1/2},\qquad R_{\alpha\beta\gamma\eta}R^{\alpha\beta\gamma\eta}(-g)^{1/2},$$
$$\text{and}\quad F_{\mu\nu}F^{\mu\nu}(-g)^{1/2} \tag{1.1}$$

are. $R^{\alpha}_{\beta\gamma\eta}$ is a Riemann tensor and $F_{\mu\nu}$ is an electromagnetic field tensor. The metric tensor has the determinant $\det|g_{\alpha\beta}|$. It has been shown that the first three scalar densities in (1.1) are interrelated because the following Hamiltonian derivatives vanish identically (Bach, 1921):

$$\frac{\partial}{\partial g^{\rho\sigma}}\left(R^2-4R_{\alpha\beta}R^{\alpha\beta}+R_{\alpha\beta\gamma\eta}R^{\alpha\beta\gamma\eta}\right)=0 \tag{1.2}$$

2. THE LAGRANGIANS R^2 AND $F_{\mu\nu}F^{\mu\nu}$

As a first case we consider the action integral

$$W=\int\left(R^2+k_0F_{\mu\nu}F^{\mu\nu}\right)(-g)^{1/2}\,dx^4 \tag{2.1}$$

0020-7748/80/1200-0873$03.00/0

where k_0 is a constant. The variations $\delta g^{\alpha\sigma}$ give the field equations (Wynne and Derrick, 1973, Eddington, 1924)

$$2g_{\rho\sigma}g^{\alpha\sigma}\mathrm{R}_{;\alpha\tau}-\mathrm{R}_{;\rho\sigma}-\mathrm{R}_{;\sigma\rho}+\frac{1}{2}g_{\rho\sigma}\mathrm{R}^2-2\mathrm{R}\mathrm{R}_{\rho\sigma}$$

$$+2k_0\left(-\frac{1}{4}g_{\rho\sigma}\mathrm{F}^{\alpha\tau}\mathrm{F}_{\alpha\tau}+\mathrm{F}^{\mu}{}_{\sigma}\mathrm{F}_{\mu\rho}\right)=0 \qquad (2.2)$$

The covariant derivatives are indicated by a semicolon subscript. The electromagnetic energy-momentum tensor is now

$$\mathrm{T}_{\rho\sigma}{}^{(EM)}=\frac{1}{2}g_{\rho\sigma}\mathrm{F}^{\alpha\tau}\mathrm{F}_{\alpha\tau}-\mathrm{F}^{\mu}{}_{\sigma}\mathrm{F}_{\mu\rho} \qquad (2.3)$$

The trace $\mathrm{T}^{\sigma}{}_{\sigma}{}^{(EM)}$ is identically zero. We then obtain from (2.2)

$$6g^{\alpha\tau}\mathrm{R}_{;\alpha\tau}=0 \qquad (2.4)$$

Assuming the curvature invariant to be constant $\mathrm{R}=4\Lambda\neq0$ (Buchdahl, 1962; Bicknell, 1974) where Λ is the cosmological constant, (2.2) takes the form

$$4\Lambda(2g_{\rho\sigma}\Lambda-2\mathrm{R}_{\rho\sigma})-2k_0\mathrm{T}_{\rho\sigma}{}^{(EM)}=0 \qquad (2.5)$$

Let us set $k_0=32\pi\Lambda$. This leads to Einstein's field equations with the cosmological constant

$$\mathrm{R}_{\rho\sigma}-\Lambda g_{\rho\sigma}=-8\pi\mathrm{T}_{\rho\sigma}{}^{(EM)} \qquad (2.6)$$

In a region free from charged particles the variations with respect to the four-potentials implicitly included in (2.1) give the Maxwell equations

$$\frac{\partial}{\partial x^{\nu}}\left[\mathrm{F}^{\mu\nu}(-g)^{1/2}\right]=0 \qquad (2.7)$$

where

$$\mathrm{F}^{\mu\nu}=\frac{\partial}{\partial x^{\nu}}\mathrm{A}^{\mu}-\frac{\partial}{\partial x^{\mu}}\mathrm{A}^{\nu}$$

is a Maxwell's tensor, which in turn satisfies

$$\frac{\partial}{\partial x^{\sigma}}\mathrm{F}_{\mu\nu}+\frac{\partial}{\partial x^{\mu}}\mathrm{F}_{\nu\sigma}+\frac{\partial}{\partial x^{\nu}}\mathrm{F}_{\sigma\mu}=0 \qquad (2.8)$$

It is well known that the Reissner–Nordström–de Sitter metric (Lake, 1979)

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2} - \frac{\Lambda}{3} r^2\right)^{-1} dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2$$

$$+\left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2} - \frac{\Lambda}{3} r^2\right) dt^2 \qquad (2.9)$$

is a special solution of (2.6), where m is the mass and Q the charge of the body.

3. THE OTHER LAGRANGIANS

Let us consider another case. The Euler–Lagrange equations corresponding to the quadratic Lagrangian $R_{\alpha\beta}R^{\alpha\beta} + k_1 F_{\mu\nu}F^{\mu\nu}$ are

$$\overline{G}_{\rho\sigma} = g_{\rho\sigma} R^{\alpha\tau}{}_{;\alpha\tau} + g^{\alpha\tau} R_{\rho\sigma;\alpha\tau} - R^{\alpha}{}_{\sigma;\rho\alpha} - R^{\alpha}{}_{\rho;\sigma\alpha} + \tfrac{1}{2} g_{\rho\sigma} R_{\alpha\tau} R^{\alpha\tau} - 2 R_{\alpha\rho} R^{\alpha}{}_{\sigma}$$

$$+ 2k_1\left(-\tfrac{1}{4} g_{\rho\sigma} F^{\alpha\tau} F_{\alpha\tau} + F^{\mu}{}_{\sigma} F_{\mu\rho}\right) = 0 \qquad (3.1)$$

As before, the trace vanishes:

$$2 g^{\alpha\tau} R_{;\alpha\tau} = 2 k_1 T^{\sigma}{}_{\sigma}{}^{(EM)} = 0 \qquad (3.2)$$

Let us suppose a spherically symmetric, static metric of the form

$$ds^2 = -\gamma^{-1} dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + \gamma\, dt^2 \qquad (3.3)$$

where $\gamma = (1 + a/r + b/r^2 + cr^2)$ includes the constants a, b, and c. Multiplying (3.1) by g^{19} and substituting the metric (3.3) in (3.1) we find[1]

$$\overline{G}^{1}{}_{1} = \frac{\gamma^2}{4}\left[\frac{2\nu'}{r}(-\nu'' - \nu'^2) + \frac{8\nu''}{r^2} + \frac{12\nu'}{r^3}\right] + \gamma \frac{\nu'}{r^3} - 36c^2$$

$$+ \frac{18\gamma}{r}\nu' c - \frac{12c}{r^2} - k_1 \frac{Q^2}{r^4} = 0 \qquad (3.4)$$

with $\nu = \ln\gamma$ and $c = -\Lambda/3 \neq 0$ for $R = 4\Lambda$.

[1] ν' denotes $(d/dr)\ln\gamma$ and so on.

Now, the equation (3.4) becomes

$$\bar{G}^1{}_1 = \frac{4b^2}{r^8} + \frac{4ab}{r^7} + \frac{4b}{r^6} - \frac{8bc}{r^4} - k_1\frac{Q^2}{r^4} = 0 \tag{3.5}$$

This shows that $b = -k_1Q^2/8c = 0$. Consequently, the Reissner–Nordström–de Sitter metric does not satisfy (3.1).

We can see from (1.2) that the field equation corresponding to the quadratic Lagrangian

$$L_1 = R_{\alpha\beta\gamma\eta}R^{\alpha\beta\gamma\eta} \tag{3.6}$$

may be written as a special case of the linear combination

$$L_2 = R_{\alpha\beta}R^{\alpha\beta} + \beta R^2 \tag{3.7}$$

if $\beta = -\frac{1}{4}$. It is therefore sufficient to consider the Lagrangian

$$L_3 = R_{\alpha\beta}R^{\alpha\beta} + \beta R^2 + k_2F_{\mu\nu}F^{\mu\nu} \tag{3.8}$$

Obviously, this Lagrangian does not have the Reissner–Nordström–de Sitter metric as a solution.

ACKNOWLEDGMENTS

The author is grateful to Professor K. Mansikka and Professor E. Suoninen for reading the manuscript.

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